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EQUATIONS OF MOTION OF NEMATIC LIQUID-CRYSTAL MEDIA

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Equations of motion for nematic liquid-crystal media in a magnetic field and also the equations of thermal conductivity are obtained. Together with the condition of continuity and the equation of state, these relationships determine the fields of nine quantities which characterize the nematic fluid: densities, pressures, temperatures, basis vectors of the local axis of anisotropy, rates of collective rotations of molecules near their "long" axes, and vectors of translational velocity. Initial conditions and boundary conditions are formulated. Special cases are examined: equilibrium of the medium in a homogeneous magnetic and temperature field, disinclinations, orientational boundary layer, and also the flow in a flat capillary in a magnetic field and the drag of fluid by a rotating magnetic field. Based on obtained results, an explanation is given for a number of effects which have been discovered experimentally earlier.

Liquid crystals occupy on the thermodynamic scale of states an intermediate (mesophase) position between anisotropic crystals and isotropic liquids. Two fundamental varieties of mesophases exist: the smectic and the nematic. In the liquid crystal medium of the smectic type the one-dimensional long-range coordination structure is preserved. The molecules are organized in regularly spaced parallel monolayers. In the medium of the nematic type the long-range order is completely absent in the spatial arrangement of molecules, just as in the ordinary liquid. However, in contrast to a liquid and in similarity to a solid crystal the long-range order of orientation for the "long" molecular axes is preserved. The orientational order is characterized in each point of the medium by the axis of mean molecular orientation. This axis is simultaneously the local axis of symmetry of the medium.

In their mechanical properties the nematic media are quite close to liquids. Experiments show [1, 2] that the behavior of nematic liquids in a force field, a temperature field, a magnetic field, and an electrical field has a number of anomalies (anisotropy of viscosity, scale effect, orientation in hydrodynamic flow, drag of the medium by a rotating magnetic field, and others.)

The peculiar combination of mechanical properties makes liquid crystal media interesting objects for investigation from the point of view of continuum mechanics. At the present time the hydrostatic theory [3 - 9] is the most developed. In papers [10 - 12] linear hydrostatics is examined with consideration of thermal conductivity and effects of rotational viscosity. The hydrodynamic theory which takes into account elastic and thermal effects in a magnetic field is just being developed [13, 14]. Some results are available in paper [9].

Liquid crystals belong to so-called media with moments or media with rotational degrees of freedom [15]. At the phenomenological level these degress of freedom are taken into account in asymmetric mechanics of continuous media. On the basis of ideas of asymmetric mechanics [16, 17] the general hydrodynamic theory for nematic liquids is developed in [13, 14] taking into account elastic, thermal and magnetic effects. Based on this development, the goal of the present paper is to obtain a closed system of equations of motion, to formulate boundary conditions and initial conditions, to clarify the most essential characteristics of the equations, and to examine the simplest cases of motion of nematic liquids.

The conservation laws for momentum and characteristic angular momentum, for mass and local moment of inertia, and also the equation for entropy balance have the form

$$\rho v_i^{\cdot} = \frac{\partial \sigma_{in}}{\partial x_n} + \rho f_i, \qquad \rho (I_{in}\alpha_n)^{\cdot} = \frac{\partial \mu_{in}}{\partial x_n} - \sigma_{nm}\,\epsilon_{imn} + \rho m_i \tag{0.1}$$

$$\rho^{\cdot} = -\rho \operatorname{div} \mathbf{v}, \qquad I_{in}^{\cdot} - (\delta_{ri}\,\epsilon_{mpn} + \delta_{rn}\,\epsilon_{mpi})\, w_m I_{pr} = 0 \tag{0.2}$$

$$(\rho s)^{\cdot} + \operatorname{div}(\mathbf{q}/T) = \Theta, \qquad \Theta \geqslant 0 \tag{0.3}$$

Here σ_{in} and μ_{in} are unsymmetric tensors of stresses and moment stresses, ρf_i and ρm_i are bulk densities of external forces and moments, v_i is the velocity of translational motion of a small region of the medium, α_i is the angular velocity of the self-rotation in this region, I_{in} is the local moment of inertia, ρ, T are, respectively, the density of the medium and the absolute temperature, ϵ_{inm} is the tensor of Levi-Civita, w_i is the angular velocity of rotation of the trihedral of the principal axes of the tensor I_{nm}, s is the entropy per unit mass, q_i is the heat flux, Θ is the entropy production in the irreversible process. The entropy production is considered as a given function. The dot indicates a substantive derivative.

For a nematic medium the local moment of inertia I_{nm} is expressed [13] through the principal values of the tensor of inertia of the molecule (i_1, i_2, i_3), the mass of the molecule m, the structural parameter j which characterizes the local degree of orientation of long molecular axes, and through the basis vector L_i of the axis of nematic molecular order (which is equivalent to the axis of anisotropy of the medium)

$$I_{nm} = I_{\perp}\delta_{nm} + (I_{\parallel} - I_{\perp})\, L_n L_m$$

$$I_{\perp} = \frac{i_1 + i_2}{2m} + \frac{2i_3 - i_2 - i_1}{6m}\,(1 - j), \qquad I_{\parallel} = \frac{i_3 + i_2}{2m} + \frac{2i_3 - i_2 - i_1}{6m}\,(1 + 2j) \tag{0.4}$$

The angular velocity of the self-rotation α_i for a nematic medium has the form

$$\alpha_i = w_i + L_i\psi, \qquad w_i = L_n L_m^{\cdot}\, \epsilon_{inm}, \qquad \psi = \lim_{\Delta N \to dN} \frac{1}{\Delta N} \sum_{\alpha=1}^{\Delta N} \psi^{\alpha} \tag{0.5}$$

Here w_i is the velocity of rotation of the axis of anisotropy L, which is simultaneously the principal axis of the tensor I_{nm}; $L_i\psi$ is the average rate of rotation of molecules around their central axes, parallel to the axis L; ψ^{α} is the molecular angular velocity; ΔN is the number of molecules in the small region of the medium. If the conditions $I_{\perp}^{\cdot} = I_{\parallel}^{\cdot} = 0$ and the first of relationships (0.4) are satisfied, as will be assumed in the following, then the law of conservation of local moment of inertia is identically satisfied.

We proceed now to the determining relationships obtained in paper [13]. Elastic phenomena in nematic media which are connected with a change of specific volume and the appearance of gradients of directions of the axis L, are described by the laws of elasticity

$$p = p^\circ + \alpha(\vartheta - \vartheta^\circ) \tag{0.6}$$

$$T_{in} = -p\delta_{in} - d_{1122}R_{ni}R_{mm} - d_{1212}R_{\alpha n}R_{\alpha i} - \\ - d_{1221}R_{n\alpha}R_{\alpha i} - (d_{1313} - d_{1212})\, R_{\alpha\beta}R_{\alpha i}L_\beta L_n \tag{0.7}$$

$$M_{in} = M_{in}^\circ + d_{1212}(R_{in} - R_{i\alpha}L_\alpha L_n) + d_{1122}(\delta_{in} - L_iL_n)\,R_{mm} + \\ + d_{1221}(R_{ni} - R_{n\alpha}L_\alpha L_i) + d_{1313}\,R_{i\alpha}L_\alpha L_n, \quad R_{in} = L_\alpha \frac{\partial L_\beta}{\partial x_n}\,\epsilon_{i\alpha\beta}, \quad R_{in}^\circ = 0 \tag{0.8}$$

Here p is the pressure, ϑ is the specific volume, T_{in} and M_{ni} are elastic stresses and elastic moment stresses, R_{in} is the tensor of orientation gradients. The superscript zero indicates values of quantities in the undeformed state. Temperature dependent material coefficients α and d_{1122}, d_{1212}, d_{1221}, d_{1313} represent the compressibility of the medium and the elasticity of the continuum of directions.

The irreversible processes of internal friction are connected with viscous flow, the local rotation of the axis of anisotropy L with respect to the surrounding region of the medium, and also with molecular rotation around the axes parallel to L. The dissipation function and the corollary rheological relationships [13] have the form

$$T\Theta = -q_iT^{-1}\,\partial T/\partial x_i + \Pi_{(in)}\,v_{(in)} + \Pi_{[in]}\,(\omega - \alpha)_m\,\epsilon_{imn} + N_\alpha\,n_\alpha \tag{0.9}$$

$$\Pi_{(in)} = a_1v_{(in)} + {}^1\!/\!_2\,a_2\,(v_{(i\alpha)}\,L_\alpha L_n + v_{(\alpha n)}\,L_\alpha L_i) + \\ + (a_3L_iL_n + a_4\delta_{in})\,L_\alpha L_\beta\,v_{(\alpha\beta)} + (a_4L_iL_n + a_5\delta_{in})\,v_{(mm)} + \\ + {}^1\!/\!_2\,a_6\,(L_nL_\alpha\epsilon_{mi\alpha} + L_iL_\alpha\,\epsilon_{mn\alpha})\,(\omega - \alpha)_m \tag{0.10}$$

$$\Pi_{[in]} = {}^1\!/\!_2\,a_6\,(v_{(n\alpha)}\,L_\alpha L_i - v_{(i\alpha)}\,L_\alpha L_n) - \\ - {}^1\!/\!_2\,[2a_7\,\epsilon_{min} + a_8\,(L_nL_\alpha\,\epsilon_{mi\alpha} + L_iL_\beta\,\epsilon_{m\beta n})]\,(\omega - \alpha)_m \tag{0.11}$$

$$N_i = (a_9\delta_{i\alpha} + a_{10}L_iL_\alpha)\,n_\alpha - bT^{-1}L_\alpha\,\partial T/\partial x_\beta\,\epsilon_{i\alpha\beta} \tag{0.12}$$

$$n_i = L_n\partial\alpha_n/\partial x_i; \quad \omega_i = {}^1\!/\!_2\,\mathrm{rot}_i\,\mathbf{v}, \quad v_{(in)} = {}^1\!/\!_2\,(\partial v_i/\partial x_n + \partial v_n/\partial x_i)$$

Here $\Pi_{(in)}$ and $\Pi_{[in]}$ are the symmetric and antisymmetric parts of the viscous stress tensor, L_iN_n is the tensor of viscous moment stresses, $v_{(in)}$ is the symmetric tensor of velocity gradients. Material coefficients from a_1 to a_8 are responsible for the bulk, shear and rotational viscosity, while a_9 and a_{10} are coefficients of "moment" viscosity; b is the gyrothermal coefficient which characterizes the appearance of viscous moment stresses in the field of temperature gradients. The complete stress and moment stress tensor is the sum of corresponding elastic and viscous components

$$\sigma_{in} = T_{in} + \Pi_{in}; \qquad \mu_{in} = M_{in} + L_iN_n$$

Irreversible processes in liquid crystals are also connected with thermal conductivity. The thermal conductivity law has the form [13]

$$q_i = -[\lambda_\perp\,\delta_{in} + (\lambda_\parallel - \lambda_\perp)\,L_iL_n]\,\partial T/\partial x_n + bL_n\,n_m\,\epsilon_{inm} \tag{0.13}$$

Here $\lambda_\parallel$ and $\lambda_\perp$ are coefficients of thermal conductivity in the parallel and perpendicular directions to

All material coefficients in (0.10) - (0.13) depend on temperature. They can also depend on the intensity of the magnetic field. However, it is possible to show that this dependence can be neglected (because of weak magnetization ability of liquid crystals).

1. Equations of motion in the kinematic form and the equation of thermal conductivity. Let us now turn to the derivation of the equations of motion. It is assumed that in (0.8) initial moment stresses are absent i.e. $M_{in}^{\circ} = 0$. This occurs when it turns out that for homogeneous orientation of directions L in space the molecular axes are also strictly parallel.

Equations of motion of a nematic medium can be obtained if the laws of elasticity (0.6) and (0.7) and the rheological laws (0.10) - (0.12) are substituted into (0.1) taking into account the relationship (0.4). Thus, substituting (0.10) and (0.11) into the first equation (0.1), we obtain the equations which describe the motion of the continuum of centers of inertia

$$\begin{aligned}\rho \mathbf{v}^{\cdot} = \rho \mathbf{f} &- \operatorname{grad} p_+ + {}^1/_2 (a_1 + 2a_5 - a_7) \operatorname{grad} \operatorname{div} \mathbf{v} + {}^1/_2 (a_1 + a_7) \Delta \mathbf{v} - \mathbf{M} \times \\ &\times \operatorname{rot} \mathbf{L} - (\mathbf{M}\nabla) \mathbf{L} + a_3 \{(\mathbf{L}\mathbf{V})(\mathbf{L}\nabla)\mathbf{L} + \mathbf{L} \operatorname{div} [\mathbf{L}(\mathbf{L}\mathbf{V})]\} + \\ &+ a_4 \{\operatorname{grad}(\mathbf{L}\mathbf{V}) + (\mathbf{L}\nabla)\mathbf{L} \operatorname{div} \mathbf{v} + \mathbf{L} \operatorname{div}(\mathbf{L} \operatorname{div} \mathbf{v})\} + \\ &+ a_6 \{(\mathbf{L}^{\cdot}\nabla)\mathbf{L} + \mathbf{L} \operatorname{div} \mathbf{L}^{\cdot} - (\mathbf{L}\nabla)\mathbf{V} - \mathbf{V} \operatorname{div} \mathbf{L}\} + a_7 \operatorname{rot}(\mathbf{L}\psi) + \\ &+ {}^1/_4 (a_2 + a_6) \{\mathbf{D} \operatorname{div} \mathbf{L} + (\mathbf{L}\nabla)\mathbf{D} + (\mathbf{D}\nabla)\mathbf{L} + \mathbf{L} \operatorname{div} \mathbf{D}\} - \\ &- {}^1/_4 (a_8 + a_6) \operatorname{rot} [(\mathbf{L} \times \operatorname{rot} \mathbf{v}) \times \mathbf{L}] + {}^1/_2 (a_8 + a_7 - a_9) \operatorname{rot} (\mathbf{L} \times \mathbf{L}^{\cdot}) \\ &(p_+ = p + {}^1/_2 M_{\alpha\beta} R_{\alpha\beta},\ \mathbf{V} = (\mathbf{L}\nabla)\mathbf{v},\ \mathbf{D} = 2\mathbf{V} + \mathbf{L} \times \operatorname{rot} \mathbf{v})\end{aligned} \tag{1.1}$$

Further, substituting (0.6), (0.7), (0.12) and also (0.4) into the second equation (0.1), we obtain the equations of motion for the continuum of directions

$$\begin{aligned}\rho [I_\perp (\mathbf{L} \times \mathbf{L}^{\cdot\cdot} + I_\parallel (\mathbf{L}^{\cdot}\psi + \mathbf{L}\psi^{\cdot})] = \rho \mathbf{m} + \mathbf{L} \times \mathbf{M} + {}^1/_2 a_6 \mathbf{D} \times \mathbf{L} + \\ + (a_8 + 2a_7) [\omega - \mathbf{L} \times \mathbf{L}^{\cdot} - \mathbf{L}\psi] + a_8 \mathbf{L} (\psi - \mathbf{L}\omega) + a_9 [(\mathbf{n}\nabla)\mathbf{L} + \mathbf{L} \operatorname{div} \mathbf{n}] + \\ + a_{10} [(\mathbf{L}\mathbf{n})(\mathbf{L}\nabla)\mathbf{L} + \mathbf{L} \operatorname{div} \mathbf{L}(\mathbf{L}\mathbf{n})] - b \{[(\mathbf{L} \times T^{-1} \operatorname{grad} T)\nabla]\mathbf{L} + \\ + \mathbf{L} \operatorname{div} (\mathbf{L} \times T^{-1} \operatorname{grad} T)\}\end{aligned} \tag{1.2}$$

In equations (1.1) and (1.2)

$$\begin{aligned}\mathbf{M} = d_{1313} \Delta \mathbf{L} - (d_{1313} - d_{1212}) \operatorname{grad} \operatorname{div} \mathbf{L} + \\ + (d_{1313} - d_{1122} - d_{1212} - d_{1221}) \{(\mathbf{L} \operatorname{rot} \mathbf{L}) \operatorname{rot} \mathbf{L} + \operatorname{rot} [\mathbf{L} (\mathbf{L} \operatorname{rot} \mathbf{L})]\}\end{aligned} \tag{1.3}$$

The first four terms on the right side in (1.1) are the same as in the Navier-Stokes equation for the ordinary liquid. Equation (1.2) degenerates for the ordinary liquid into the following relationship

$$(a_8 + 2a_7) [\omega - \mathbf{L} \times \mathbf{L}^{\cdot} - \mathbf{L}\psi] + a_8 \mathbf{L} (\psi - \mathbf{L}\omega) = 0$$

which is equivalent to the trivial relationship $\omega = \mathbf{L} \times \mathbf{L}^{\cdot} + \mathbf{L}\psi$.

Equations (1.1) and (1.2) form a system of six equations for nine unknown functions: v_i, L_i, ψ, ρ, p and T. In order to obtain a closed system of equations, it is necessary to add to Eqs. (1.1) and (1.2) the equation of conservation of mass (0.2), the relationship (0.6), and the equation of thermal conductivity. The equation of thermal conductivity is obtained by substituting the expression for the dissipation function (0.9) into the equation for entropy balance (0.3). Taking into account the law of thermal conductivity (0.13), we obtain

$$\begin{aligned}\rho T s^{\cdot} = \frac{\partial}{\partial x_i} \left\{ [\lambda_\perp \delta_{in} + (\lambda_\parallel - \lambda_\perp) L_i L_n] \frac{\partial T}{\partial x_n} + b L_\alpha n_\beta \epsilon_{i\alpha\beta} \right\} + \\ + \Pi_{(in)} v_{(in)} + \Pi_{[in]} (\omega - \mathbf{a})_m \epsilon_{imn} + N_\alpha n_\alpha\end{aligned} \tag{1.4}$$

It is evident from (1.4) that the change in entropy is connected with thermal conductivity and dissipation of mechanical energy in the motion of centers of inertia of molecules and during their rotation. If

$$T = \text{const}, \; v_i = \text{const}, \; L^{\cdot} = 0, \; \psi = 0$$

then it follows from (1.4) that $s^{\cdot} = 0$. In the general case $s = s(T, \vartheta, M_i, R_{in})$ (M_i is the magnetization of the medium). Therefore,

$$s^{\cdot} = T^{\cdot}\left(\frac{\partial s}{\partial T}\right)_{\vartheta, R, M} + \vartheta^{\cdot}\left(\frac{\partial s}{\partial \vartheta}\right)_{T, R, M} + M_i^{\cdot}\left(\frac{\partial s}{\partial M_i}\right)_{T, R, \vartheta} + R^{\cdot}_{in}\left(\frac{\partial s}{\partial R_{in}}\right)_{T, \vartheta, M}$$

The last two terms on the right side describe the magnetocaloric and the orientational gradient thermal effects. Both of these are insignificantly small compared to the first two because of the quite small magnetization of liquid crystals and the small energy of thermodynamic transition of a nematic liquid crystal into an isotropic liquid. Consequently,

$$s^{\cdot} \simeq T^{\cdot}\left(\frac{\partial s}{\partial T}\right)_{\vartheta} + \vartheta^{\cdot}\left(\frac{\partial s}{\partial \vartheta}\right)_{T}$$

By virtue of the thermodynamic identity $(\partial s / \partial \vartheta)_T \equiv (\partial p / \partial T)_\vartheta$ we obtain

$$s^{\cdot} \approx T^{-1}T^{\cdot}c_\vartheta - \vartheta^{\cdot}(\partial p / \partial T)_\vartheta$$

If we neglect the effect of orientation gradients on the specific volume (the effect is of second order in smallness with respect to $\partial L_i / \partial x_n$), we can write

$$\vartheta^{\cdot} = T^{\cdot}\left(\frac{\partial \vartheta}{\partial T}\right)_p, \quad s^{\cdot} = T^{-1}T^{\cdot}c_p, \qquad c_p = c_\vartheta - T\left(\frac{\partial p}{\partial T}\right)_\vartheta \left(\frac{\partial \vartheta}{\partial T}\right)_p \tag{1.5}$$

Here c_p is the specific heat at constant pressure. Taking in account (0.10) - (0.12) and (1.5), we obtain

$$\begin{aligned} c_p T^{\cdot} &= \lambda_\perp \Delta T + (\lambda_\parallel - \lambda_\perp)\,[L \operatorname{grad} T \operatorname{div} \mathbf{L} + \mathbf{L} \operatorname{grad}(\mathbf{L} \operatorname{grad} T)] + \\ &+ b \operatorname{grad} \psi \operatorname{rot} \mathbf{L} + a_1 v_{(nm)} v_{(nm)} + a_2 L_i v_{(in)} L_m v_{(mn)} + a_3 (L_i L_n v_{(in)})^2 + \\ &+ 2a_4 (L_i L_n v_{(in)}) v_{(mm)} + a_5 v_{(nn)} v_{(mm)} + 2a_6 (L_i v_{(in)} L_q \epsilon_{mnq})(\omega - \alpha)_m + \\ &+ 2a_7 (\omega - \alpha)_n (\omega - \alpha)_n + a_8 \{(\omega - \alpha)^2 - [\mathbf{L}(\omega - \alpha)]^2\} + a_9 n_i n_i + a_{10} (L_i n_i)^2 \end{aligned} \tag{1.6}$$

This is the thermal conductivity equation generalized to the case of motion of an anisotropic nematic medium. If $\lambda_\perp = \lambda_\parallel = \lambda$, and the medium is at rest $(v_i = 0, \alpha_i = 0)$, Eq. (1.6) transforms into the usual equation of thermal conductivity of an isotropic body. Equation (1.6) represents the relationship which must be added to (1.1) and (1.2), the law of mass conservation (0.2), and the relationship (0.6) in order to obtain a closed system of equations.

Equations (1.1) and (1.2) are substantially simplified if the medium is at rest ($v_i = 0$, $\alpha_i = 0$). They assume the following form

$$\rho \mathbf{f} = \operatorname{grad} p_+ - \mathbf{M} \times \operatorname{rot} \mathbf{L} - (\mathbf{M}\nabla)\mathbf{L} \tag{1.7}$$

$$\rho \mathbf{m} = \mathbf{M} \times \mathbf{L} \tag{1.8}$$

The system (1.7), (1.8) describes the statics of nematic liquid crystal media. This is a system of six equations for three functions: pressure p and basis vector L_i. Nevertheless, the system (1.7), (1.8) turns out to be compatible for some restrictions placed on ρf_i and ρm_i. In fact, if we form the convolution of vector ρm_i with the tensor R_{in} and combine the resulting equations with (1.7), we obtain

$$\rho f_i + \rho m_\alpha R_{\alpha i} = \text{grad}\, p_+ \tag{1.9}$$

i.e., the system (1.7), (1.8) turns out to be equivalent to system (1.8), (1.9). The latter will be solvable if

$$m_i L_i = 0, \qquad \rho f_i + \rho m_\alpha R_{\alpha i} = \text{grad}_i\, U \tag{1.10}$$

This is a necessary condition of equilibrium for a nematic liquid crystal medium.

The most important case is the equilibrium under the action of bulk forces which have the potential

$$\rho \mathbf{f} = -\,\text{grad}\, g$$

and of bulk moments which originate in the homogeneous magnetic field H. Then, due to the anisotropy in magnetic susceptibility $\Delta\chi = \chi_\parallel - \chi_\perp \neq 0$ ($\chi_\parallel$ and $\chi_\perp$ are the magnetic susceptibilities in the directions parallel and perpendicular to L) we have

$$\rho \mathbf{m} = \Delta\chi\,(\mathbf{LH})\,[\mathbf{L}\times\mathbf{H}] \tag{1.11}$$

It can be shown that for this case conditions (1.10) are satisfied and here

$$U = -g - {}^1\!/_2\,\Delta\chi\,(\mathbf{LH})^2, \qquad g + {}^1\!/_2\,\Delta\chi\,(\mathbf{LH})^2 + p_+ = \text{const}$$

The last equation represents the integral of equations (1.9).

In this manner, if conditions (1.10) are applicable, the solution of the equations for statics of nematic liquid crystal media is in essence reduced to finding the solution of Eq. (1.8) which agrees with the equations of Oseen [3, 5].

2. Initial conditions and boundary conditions. Since Eqs. (1.1), (1.2), (1.6) and (0.2) contain the first derivative with respect to time for the translational velocity v_i, the angular velocity ψ, the density ρ and the temperature T, and the second derivative of the basis vector of the primary orientation of molecules L_i, the following quantities must be given at the initial instant

$$v_i(\mathbf{r},0), \quad \psi(\mathbf{r},0), \quad w_i(\mathbf{r},0), \quad \rho(\mathbf{r},0), \quad T(\mathbf{r},0) \tag{2.1}$$

and, in addition, the field of the basis vector L_i, i.e. the initial orientational structure of the medium $L_i(\mathbf{r}, 0)$. Here the case with initial homogeneous orientation is examined, i.e. $L_i(\mathbf{r}, 0) = \text{const}$. For the vector of translational velocity v_i the kinematic boundary conditions are most simply realized. According to the hypothesis of adhesion to the solid surface which is impenetrable to the liquid, we have

$$v_i(\mathbf{r},t)\big|_\sigma = v_i^\sigma(t) \tag{2.2}$$

Here v_i^σ is the velocity of the boundary surface. The definition of the vector L_i on the surface σ is in practice readily attainable through special treatment of the boundary surface. For this reason the following boundary conditions are physically justifiable

$$L_i(\mathbf{r},t)\big|_\sigma = L_i^\sigma(t) \tag{2.3}$$

The meaning of the boundary condition for ψ is physically not clear due to the lack of knowledge about the interaction of molecules, which rotate with respect to the long axes in the medium, and the solid wall. For the formulation of this condition we therefore resort just as in [17] to the hypothesis of rotational friction of molecules against the rigid wall. Assuming that this stipulates the dissipative moment stresses $L_i N_n$ on the boundary σ, we have

$$L_i N_m \nu_m |_\sigma = \beta_{im} L_m (\psi - \mathbf{L}\omega) |_\sigma \tag{2.4}$$

Here β_{im} is the symmetric tensor of rotational surface friction which characterizes the interaction of the medium with the rigid boundary, ν_m is the external normal to the boundary surface σ. Remembering that for the tensor $L_i N_n$ only the L-component is different from zero and that by virtue of (0.12) N_n can be represented by n_i, we finally write

$$a_9 (\mathbf{\nu n}) + a_{10} (\mathbf{\nu L}) (\mathbf{Ln}) - \beta\psi |_\sigma = b T^{-1} (\mathbf{\nu} \times \mathbf{L}) \operatorname{grad} T - \beta \mathbf{L}\omega |_\sigma \tag{2.5}$$

Here the constant $\beta = \beta_{im} L_i L_m$ must be found experimentally. The boundary conditions for the temperature are formulated in the theory of thermal conductivity [18]: on the boundary the temperature or the heat flux must be given (mixed boundary conditions are also possible). The formulated initial conditions and boundary conditions allow to determine nine functions v_i, L_i, ρ, p, ψ and T from nine equations (1.1), (1.2), (1.6), (0.6) and from the first equation (0.2).

Since the finding of general solutions for the obtained equations represents a difficult task, it is appropriate to examine simple cases of motion. Analysis of these cases permits not only to elucidate the most important characteristics of the geneneral equations, but also to make a comparison with results of experimental studies of mechanical behavior of nematic media.

3. Disinclinations. If $\rho f_i = 0$ and $\rho m_i = 0$, the condition of statics (1.10) is satisfied, and the equilibrium of a nematic medium is described by Eq. (1.8) taking into account (1.3). This equation is satisfied by $L_i = \text{const}$, i.e. uniform orientation of axes L. However, disruptions of the homogeneous orientation are possible. This occurs for example if a microadditive gets in to the liquid crystal medium, and an inhomogeneous field of L directions is formed around this additive. When motion occurs, this additive is carried along by the moving front of crystallization and leaves a trace, a line of singularity of the field L which is called disinclination.

Limiting ourselves to the case of plane deformation of axes L we assume that

$$L_r = \cos\Phi, \qquad L_\varphi = \sin\Phi, \qquad L_z = 0, \; \Phi = \Phi(\varphi) \tag{3.1}$$

Taking into account (3.1) in Eqs. (1.8) and (1.3) we obtain

$$\frac{d}{d\varphi}\left\{(1 - d_* \cos 2\Phi)\left(\frac{d\Phi}{d\varphi}\right)^2 + d_* \cos 2\Phi\right\} = 0, \quad d_* = \frac{d_{1313} - d_{1212}}{d_{1313} + d_{1212}} \tag{3.2}$$

The integral of this equation must satisfy the boundary conditions (3.3)

$$\Phi = \Phi_0 \quad \text{for} \quad \varphi = \varphi_0, \quad \Phi = \Phi_0 + k\pi \quad \text{for} \quad \varphi = \varphi_0 + 2\pi \quad (k = 0, \pm 1, \ldots)$$

The meaning of the first condition is apparent. The second condition is the condition of periodicity of azimuth Φ taking into account the physical indescernibility of L_i and $-L_i$. If it is assumed that $d_* = 0$, the solution of Eq. (3.2) satisfying boundary conditions (3.3) and the equation for vector field lines of L have the following form, respectively

$$\Phi = {}^1/_2\, k\varphi + \Phi_0, \qquad r = r_0 \left\{\sin\left({}^1/_2\, k\varphi - \Phi_0\right)\right\}^{2/k} \tag{3.4}$$

This case was examined by Oseen [4] and Frank [6]. The paper [19] is devoted to the analysis of the case $d_{\cdot\cdot} \neq 0$. In these papers the equations which describe disinclinations were obtained from the condition of minimum of the elastic potential. These equations agree with (3.2) which represents a special case of equations of motion (1.1), (1.2). The pictures of the configuration of the field of long molecular axes predicted by Eqs. (3.2) are in good agreement with experimentally observed pictures obtained due to disinclination in nematic media.

4. Static gradient of directions of long axes of molecules. If the liquid crystal medium is located between parallel surfaces $x = + l$ in a constant magnetic field, then per unit volume of medium of orienting bulk moment ρm_i is active. This moment is determined by the expression (1.11). Under the action of this moment a static gradient of directions of the long molecular axes arises under the condition that the orienting action of the wall hinders the turn of molecules along the magnetic field H_i. This effect was utilized in the determination of elastic moduli of the continuum of directions d_{ikmn} [20 - 22]. In these papers the authors assumed that the bulk moment ρm_i is balanced by the orienting moment of the wall which was taken to be equal to $A d^2\Phi / dx^2$, i.e.

$$A\partial^2 \Phi/\partial x^2 = \Delta\chi H^2 \sin\Phi\cos\Phi \tag{4.1}$$

Here Φ is the angle between the basis vector L_i and the vector of magnetic field intensity, A is the modulus of elasticity of the continuum of directions.

The statics of nematic media is described by Eq. (1.8) from which it is possible to obtain the equations for various types of deformations of the continuum of directions. Three kinematically independent types of deformations are recognized depending on the relative orientation of the vectors L_i and H_i : buckling, lateral flexure and torsion [21, 22]. Each type of deformation has its own equation. If

$$\mathbf{L} = \cos\Phi \mathbf{e}_x + \sin\Phi \mathbf{e}_y, \qquad \mathbf{H} = H\mathbf{e}_y, \qquad \Phi = \Phi(x)$$

then a lateral flexure of the field of directions L arises. This is characterized by the equation

$$\begin{aligned} d_{1313}\frac{d^2\Phi}{dx^2} - (d_{1313} - d_{1212})\left\{\sin^2\Phi\frac{d^2\Phi}{dx^2} + \sin\Phi\cos\Phi\left(\frac{d\Phi}{dx}\right)^2\right\} = \\ = \Delta\chi H^2\sin\Phi\cos\Phi \end{aligned} \tag{4.2}$$

If

$$\mathbf{L} = \sin\Phi\mathbf{e}_x + \cos\Phi\mathbf{e}_z, \qquad \mathbf{H} = H\mathbf{e}_x, \quad \Phi = \Phi(x)$$

then buckling takes place

$$\begin{aligned} d_{1212}\frac{d^2\Phi}{dx^2} + (d_{1313} - d_{1212})\left\{\sin^2\Phi\frac{d^2\Phi}{dx^2} + \sin\Phi\cos\Phi\left(\frac{d\Phi}{dx}\right)^2\right\} = \\ = \Delta\chi H^2\sin\Phi\cos\Phi \end{aligned} \tag{4.3}$$

Finally, if

$$\mathbf{L} = \cos\Phi\mathbf{e}_y + \sin\Phi\mathbf{e}_z, \qquad \mathbf{H} = H\mathbf{e}_z, \qquad \Phi = \Phi(x)$$

then the torsional deformation of the continuum of directions is described by the equation

$$(d_{1212} + d_{1221} + d_{1122})\frac{d^2\Phi}{dx^2} = \Delta\chi H^2\sin\Phi\cos\Phi \tag{4.4}$$

Moduli d_{1313}, d_{1212}, $d_{1212} + d_{1221} + d_{1122}$ are called moduli of elasticity of the corresponding types of deformations which are analogous to deformations of the solid body, although not identical with them [2]. Equation (4.4) agrees exactly with the empirical equation (4.1), while Eqs. (4.2) and (4.3) agree with (4.1) if $d_{1313} = d_{1212}$. Although Eqs. (4.2) and (4.3) are different from Eq. (4.1), it is possible to show that their solution for boundary conditions

$$\Phi(l) = \Phi(-l) = 0$$

which follow from (2.3) for $L^\sigma = 0$, lead to the same result as the solution of Eq. (4.1). Namely, a layer of the thickness Z_0 is deformed by the magnetic field if $H > H^*$ Here

$$z_0 H^* = \mathrm{const} \tag{4.5}$$

The constant in the right side of (4.5) is determined by moduli of elasticity and the magnetic properties of the medium. Only the character of approximation to the orientation of axes with respect to the field depens on d_*. The law (4.5) is preserved for all Eqs. (4.1) - (4.4). The relationship (4.5) has been experimentally confirmed more than once and by different methods.

In this manner we can consider that the equations of statics of the continuum of directions give the correct result for the case of turn of long molecular axes by the magnetic field.

5. Flow in a plain capillary. Let us examine the steady flow of a nematic medium in a plane capillary. The length of the capillary is a, the width is h, and the height is $2l$. We assume that $h \gg l$, then we can neglect edge effects and to examine the flow between infinite parallel planes. Let the medium in a magnetic field H_i which is perpendicular to the planes $x = \pm l$ move along the z-axis. The long axes of molecules rotate in the plane xz, i.e.

$$\mathbf{v} = v(x)\,\mathbf{e}_z, \quad \mathbf{L} = \cos\Phi\,\mathbf{e}_x + \sin\Phi\,\mathbf{e}_z, \quad \mathbf{H} = H\mathbf{e}_x, \quad \Phi = \Phi(x) \tag{5.1}$$

Furthermore we assume that $T = \mathrm{const}$, $\psi = 0$, $\rho f_i = 0$, and the bulk moment ρm_i is given by the expression (1.11). Then, substituting (5.1) into (1.1) and (1.2), we obtain the following equations for the pressure p_+, the velocity v_i and the azimuth Φ:

$$\begin{gathered}\frac{\partial p_+}{\partial x} = \frac{d}{dx}\Big[({}^1\!/_2 a_2 + a_4 + {}^1\!/_2 a_6 + a_3\cos^2\Phi)\sin\Phi\cos\Phi\,\frac{dv}{dx} - \\ - {}^1\!/_2\,(d_{1313}\cos^2\Phi + d_{1212}\sin^2\Phi)\left(\frac{d\Phi}{dx}\right)^2\Big]\end{gathered} \tag{5.2}$$

$$\begin{gathered}\frac{\partial p_+}{\partial y} = 0, \quad \frac{\partial p_+}{\partial z} = \frac{d}{dx}\left[(\eta_\perp(\infty) + a_3\sin^2\Phi\cos^2\Phi + a_6\sin^2\Phi)\frac{\partial v}{\partial x}\right] \\ \eta_\perp(\infty) = {}^1\!/_4\,(a_8 + a_2 + 2a_1 + 2a_7 - 2a_6) \\ \frac{d}{dx}\left[{}^1\!/_2\,(d_{1313}\cos^2\Phi + d_{1212}\sin^2\Phi)\left(\frac{d\Phi}{dx}\right)^2\right] + \\ + \left[({}^1\!/_2 a_8 + a_7 - {}^1\!/_2 a_6 + a_6\sin^2\Phi)\frac{dv}{dx} - \Delta\chi H^2\sin\Phi\cos\Phi\right]\frac{d\Phi}{dx} = 0\end{gathered} \tag{5.3}$$

These equations must be integrated for boundary conditions (2.2) and (2.3) in which $v^\sigma = 0$, and $\mathbf{L}^\sigma = \mathbf{e}_x$, i.e. it is assumed that on surfaces $x = \pm l$ the axes L do not rotate

$$v(l) = v(-l) = 0, \quad \Phi(l) = \Phi(-l) = 0 \tag{5.4}$$

From (5.2) we determine the pressure p_+ and $dv\,/\,dx$

$$p_+ = [{}^1/_2 a_2 + a_4 + {}^1/_2 a_6 + a_3 \cos^2 \Phi] \sin\Phi \cos\Phi \frac{dv}{dx} - \\ - {}^1/_2 [d_{1313} \cos^2\Phi + d_{1212} \sin^2\Phi] \left(\frac{d\Phi}{dx}\right)^2 + z \frac{\partial p_+}{\partial z} \\ \frac{\partial v}{\partial x} = x \frac{\partial p_+}{\partial z} [\eta_\perp(\infty) + a_3 \sin^2\Phi \cos^2\Phi + a_6 \sin^2\Phi]^{-1}, \quad \frac{\partial p_+}{\partial z} = \text{const.} \tag{5.5}$$

The constant of integration in (5.5) is equal to zero by virtue of the symmetry of the velocity field $v(x)$. Substituting $(5.5)_2$ into (5.3) we obtain a differential equation which must be satisfied by the azimuth Φ

$$\varepsilon \frac{d}{d\xi}\left\{\frac{1}{2}[\cos^2\Phi + d_{1212}/d_{1313}\sin^2\Phi]\left(\frac{d\Phi}{d\xi}\right)^2\right\} + \{\mu\xi[{}^1/_2 a_8 + a_7 - {}^1/_2 a_6 + a_6 \sin^2\Phi] \times \\ \times [\eta_\perp(\infty) + a_3 \sin^2\Phi\cos^2\Phi + a_6\sin^2\Phi]^{-1} - \sin\Phi\cos\Phi\} \frac{d\Phi}{d\xi} = 0 \tag{5.6}$$

$$\xi = x/l, \qquad \varepsilon = d_{1313}(\Delta\chi H^2 l^2)^{-1}, \qquad \mu = l\partial p_+/\partial z\,(\Delta\chi H^2)^{-1}$$

For the experimental investigation of flow of nematic media [24], capillaries with $l \sim 10^{-2}$ cm and $\partial p_+/\partial z\, 10^{-2}$ dynes/cm^3 were used. Taking into account that the moduli of elasticity d_{1313}, $d_{1212} \sim 10^{-6}$ dynes and $\Delta\chi \sim 10^{-6}$ cm^3/g, we obtain that in a magnetic field with an intensity of several thousand oersted under ordinary experimental conditions the following relations are valid $|\varepsilon| \ll 1$, $|\mu| \ll 1$ and $|\varepsilon| \ll |\mu|$. In this manner Eq. (5.6) together with conditions (5.4) for Φ represents a nonlinear boundary value problem with a small parameter associated with the highest derivative. For this boundary value problem [23] theorems of existence, of uniqueness, and of uniform tendency of the solution to the solution of the degenerate equation (for $\varepsilon = 0$) have been proved.

If we are satisfied with an accuracy of $O(\varepsilon)$, it is possible to limit oneself to the solution of the degenerate equation

$$2\alpha_1\mu\xi = \sin 2\Phi\,(\cos^2 2\Phi + \alpha_1 \cos 2\Phi - \alpha_2)(\cos 2\Phi - \alpha_3)^{-1} \tag{5.7}$$

$$\alpha_1 = 2a_6 a_3^{-1},\ \alpha_2 = 1 + [2a_6 + 4\eta_\perp(\infty)]\, a_3^{-1},\ \alpha_3 = 1 + (a_8 + 2a_7 - a_6)a_3^{-1}$$

The quantity of liquid flowing out per unit time is

$$Q = h\int_{-l}^{l} v(x)\,dx = -2hl\int_0^l \xi \frac{dv}{d\xi}\,d\xi \tag{5.8}$$

If (5.5) and (5.7) are substituted into (5.8), we obtain

$$Q = -{}^1/_4 h l^3 a_3^2 \mu^{-3} a_6^{-3} \frac{\partial p_+}{\partial z} Q_1(\varphi) \tag{5.9}$$

$$Q_1(\varphi) = B_0\varphi + \sum_{n=1}^{4} B_n \sin 2n\varphi + \sin 2\varphi \sum_{n=1}^{3} B_{-n}(\cos 2\varphi - \alpha_3)^{-n} + \\ + 2D(1-\alpha_3^2)^{-1/2}\,\mathrm{Arcth}\left(\sqrt{\frac{1+\alpha_3}{1-\alpha_3}}\,\mathrm{tg}\,\varphi\right)$$

$$B_0 = (10t + 2r - 4)\,\alpha_3 + {}^1/_4\,(3 - 2r + 10\alpha_2) - \alpha_3(\alpha_1\alpha_2 + {}^9/_2\,\alpha_1)$$
$$B_1 = 5t\alpha_3 + {}^1/_8\,[(8r-5)\,\alpha_3 - 7\alpha_1 - 4\alpha_1\alpha_2]$$
$$B_2 = t + {}^1/_8(r + q + \alpha_2),\ B_3 = {}^1/_{24}\,(3\alpha_1 + 5\alpha_3),\ B_4 = {}^1/_{32}$$
$$B_{-1} = {}^1/_{12}\,t\,(8t + 3q),\ B_{-2} = {}^1/_{12}\,t\,[(10t + 3q)\,\alpha_3 - 6\alpha_3 - 3\alpha_1]$$

$$B_{-3} = -{}^1/_6 t^2 (1 - \alpha_3^2)$$

$$D = t(3t + 2q - \alpha_2 - {}^5/_2)\alpha_3 - {}^3/_4 t\alpha_1 + {}^1/_2 (1 - \alpha_3^2) \quad (\alpha_1 + 2\alpha_3)(1 + 2\alpha_3^2 - \alpha_2)$$

$$t = \alpha_3^2 + \alpha_1\alpha_3 - \alpha_2, \; q = 2\alpha_2 - \alpha_1\alpha_3, \; r = \alpha_1^2 + 3\alpha_2 - 1$$

The parameter φ is the value of the azimuth Φ for $\xi = 1$, if we limit ourselves to the degenerate solution. Then the dependence of H on parameter φ follows from (5.7)

$$H = H_* H_1(\varphi), \quad H_* = \left[-2\frac{\partial p_+}{\partial z}\frac{l}{\Delta\chi}\right]^{1/2}$$

$$H_1(\varphi) = \frac{[\alpha_1(\cos 2\varphi - \alpha_3)]^{1/2}}{[\sin 2\varphi\,(\alpha_2 - \alpha_1\cos 2\varphi - \cos^2 2\varphi]^{1/2}} \tag{5.10}$$

Experimentally it is more convenient to investigate the time T for the outflow of a fixed volume Q° as a function of the magnetic field intensity H. It is apparent that $T = Q/ / Q^\circ$, then we obtain from (5.9)

$$T = T_\infty T_1(\varphi), \quad T_\infty = -3Q^\circ\eta_\perp(\infty)\left(2hl^3\frac{\partial p_+}{\partial z}\right)^{-1}, \quad T_1(\varphi) = \frac{Q_1(\varphi)}{24H_1^3(\varphi)} \tag{5.11}$$

It is evident that T_∞ is the time of outflow for the volume Q° of an ordinary liquid with the viscosity $\eta_\perp(\infty)$, while Eq. (5.10) together with (5.11) represents the dependence of T on H given in a parametric form.

Let us analyze the character of this dependence.

1. If $H \to \infty$, then $\varphi \to 0$, $T_1(\varphi) \to 1$, $T \to T\infty$. This means that for $H \to \infty$ the curve $T(H)$ asymptotically approaches T_∞ and the nematic medium flows as an ordinary Newtonian liquid with the viscosity $\eta_\perp(\infty)$. The physical meaning of the coefficient of viscosity $\eta_\perp(\infty)$ is clear from this.

2. If $H \to 0$, then:

$$\varphi \to {}^1/_2 \quad \arccos\alpha_3,$$

$$T_1(\varphi) \to \eta_\perp(0) / \eta_\perp(\infty), \qquad T \to T_0 = -3Q^\circ\eta_\perp(0)(2hl^3\partial p_+ / \partial z)^{-1}.$$

From this limiting relationship it is clear that for $H \to 0$ also, the nematic medium behaves as an ordinary Newtonian liquid, but it has now the following coefficient of viscosity:

$$\eta_\perp^{(0)}) = {}^1/_4 (a_2 + a_8 + 2a_1 + 2a_6 + 2a_7) + {}^1/_2 (1 + \alpha_3)a_6 [{}^1/_2 a_3 a_6^{-1} (1 - \alpha_3) - 1]$$

3. The upper asymptote T_∞ and the lower T_0 are inversely proportional to the gradient of pressure $\partial p_+ / \partial z$.

The character of dependence predicted by Eqs. (1.1) and (1.2) and the relationships (1) - (3) agree with conclusions drawn from experimental investigation of the action of the magnetic field on the rate of outflow of p-azoxyanisole from a capillary [24]. Thus, the equations obtained for the motion of nematic liquid crystal media not only predict the fact of anisotropy of viscosity, but also lead (even in the zeroth approximation) to the correct dependence of anisotropy of viscosity on the magnetic field intensity.

We note that the investigated conditions of flow correspond to the case where: a) the vector of velocity v_i is perpendicular to the vector L_i° (orientation of L-axes in the initial state), and L°_i is perpendicular to the vector $\omega = {}^1/_2\mathrm{rot}\mathbf{v}$. In an analogous manner it is possible to examine two other flow conditions: b) v_i is parallel to L_i° and L_i° is perpendicular to ω_i; c) v_i is perpendicular to L_i°, and L_i° is parallel to ω_i. These three viscosimetric flow conditions were used for experimental measurement of viscosity coefficients of nematic liquid-crystal media.

6. The effect of "drag". Let us examine the motion of a liquid-crystal medium in a long cylindrical vessel the axis oz of which is perpendicular to the direction of a homogeneous magnetic field $\mathbf{H} = H\mathbf{e}_x$. The cylinder rotates with a constant angular velocity ω. Then Eqs. (1.1) and (1.2) with the boundary condition (2.2) for velocity v_i and with the dynamic condition for L_i

$$v_i|_{r=R} = \omega_n R_m \epsilon_{inm}, \qquad \mu_{zr}|_{r=R} = 0 \tag{6.1}$$

allow the following flow mode:

$$v_i = \omega_n R_m \epsilon_{inm}, \qquad \mathbf{L} = \cos\alpha(t)\,\mathbf{e}_x + \sin\alpha(t)\,\mathbf{e}_y \tag{6.2}$$

For this mode Eqs. (1.1) and (1.2) take the form

$$\rho\mathbf{f} = \operatorname{grad} p, \quad (a_8 + 2a_7)\left(\frac{d\alpha}{dt} - \omega\right) = -\Delta\chi H^2 \sin 2\alpha \tag{6.3}$$

The second of Eqs. (6.3) was presented in papers [25-27] for explanation of regularities of the effect of "drag" on the liquid by a rotating magnetic field. We note that this equation follows from (1.1) and (1.2) for the condition that the moment stresses at the wall are equal to zero, i.e. orientational interactions of molecules with the walls are absent. Furthermore, the rate of rotation of cylinder walls ω is determined by the angle α_0 in the stationary mode of motion, i.e.

$$\omega = (a_8 + 2a_7)^{-1}\Delta\chi H^2 \sin 2\alpha_0$$

If these conditions are not satisfied, the mode of motion becomes more complicated, which in fact is observed experimentally. A complete quantitative theory of this effect requires special examination of Eqs. (1.1) and (1.2).

In conclusion, we can say that the developed hydrodynamics of nematic media explains the most important regularities which are observed experimentally. This fact apparently testifies in favor of the adequacy of obtained equations for the peculiar mechanical behavior of liquid-crystal media.

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